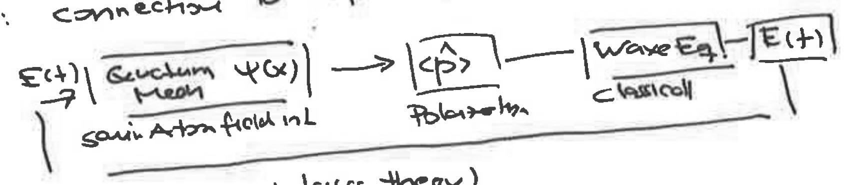
Quantum Optics Lecture 7

Semiclassical Thy.

Semiclassical atom light interaction: Maxwell–Schrödinger eq.

Aim: connection to refractive index and susceptibility.



(e.g. Lamb Laser Physics)

(→ Semiclassical laser theory)

required: introduce decay of atoms into model (ie dissipation) such as spontaneous emission.

recall: $P(z,t) = \varepsilon_0 \int_{-\infty}^{t} \chi(\tau)\, E(z,\underset{\substack{\downarrow \\ t'=t-\tau}}{t'})\, dt'\,\tau$

ie $p(\omega) = \underbrace{\chi(\omega)}_{\text{complex}} \varepsilon_0 E(\omega)$ susceptibility. Re ∧ Im gives refractive index ∧ absorption.

Electrodynamics: HELMHOLTZ EQUATION

$$\boxed{\frac{\partial^2 \vec{E}}{\partial z^2} - \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t^2} = \mu_0 \underbrace{\frac{\partial^2 \vec{P}}{\partial t^2}}_{\text{Polarization}}}$$

ATOMIC POLARIZATION:

Note that $\vec{P}(z,t) = \underbrace{\vec{p}_{12}}_{\text{Dipole matrix element}} \cdot (c_1 c_2^* + c_2 c_1^*) = \vec{p}_{12}(\rho_{12} + \rho_{12}^*)$

$$\langle 1 | q\hat{\vec{r}} | 2 \rangle = p_{12} = p_{21}$$

Note that we call $c_1 c_2^* = \rho_{12}$ ie the elements of the density matrix for the two level system.

$$\rho = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix} \qquad \rho_{11} = |c_1|^2 \quad \rho_{22} = |c_2|^2$$
$$\rho_{12} = c_1 c_2^* \quad \rho_{21} = c_2 c_1^*$$

Key point: to derive $n(\omega) = n_R + i n_{Im}$ the refractive index (and loss/gain) we need dissipation, as derived via full quantum mech. model (spontaneous emission)

From Schrödy picture $\frac{d}{dt}\rho = -\frac{i}{\hbar}[H,\rho]$: Hamiltonian evolution.

$$\dot\rho_{11} = -\frac{i}{\hbar}(\vec{p}_{12}\vec{E}) \cdot \{\rho_{12}\} + c.c \qquad + A_{12}\rho_{22}$$
$$\dot\rho_{22} = +\frac{i}{\hbar}(\vec{p}_{12}\vec{E}) \cdot \{\rho_{12}\} + c.c \qquad - A_{12}\rho_{22}$$
$$\dot\rho_{12} = -i\Delta\rho_{12} + \frac{i}{\hbar}\vec{p}_{12}\underset{\uparrow\,\vec{E}(t)}{\vec{E}}\underbrace{(\rho_{11}-\rho_{22})}_{\text{inversion.}} \qquad \underbrace{- A_{12}/2\,\rho_{12}}_{\text{DISSIPATION}}$$

$\Delta = \omega_{12} - \omega$

→ Formally (StatPhys $\underline{\text{IV}}$)

$$\dot{\hat\rho} = -\frac{i}{\hbar}[H,\hat\rho] - \frac{1}{2}\{\hat{\Gamma},\hat\rho\} \qquad \Gamma_{12} \mathrel{\hat=} A_{12} \mathrel{\hat=} \frac{1}{4\pi\varepsilon_0}\frac{4\omega^3 p_{12}}{3\hbar c^3}$$

Spont. emission

$$\Rightarrow \boxed{\dot\rho_{12} = -(i\Delta - \overset{\overset{\hat=\, i\Omega}{\times}}{\widetilde{A_{12}}})\rho_{12} + \left(\frac{i}{\hbar}\vec{p}_{12}\vec{E}\right)\cdot(\rho_{11}-\rho_{22})} \Rightarrow \rho_{12} = \frac{i p/\hbar\, \vec{p}_{12}\vec{E}}{i\Delta + A_{12}}(\rho_{11}-\rho_{22})$$

Quantum Optics (Lecture 7)

Semi-classical Thy

Scully ch. 7.5

refractive index / gain: Definition of the susceptibility

$$\boxed{P(z,t) = \varepsilon_0(\underbrace{\chi'_R + i\chi''_{img}}_{\chi}) \cdot \mathcal{E}(z,t)} \qquad \chi \hat{=} \frac{\vec{p}_{12}^{\,2}}{\varepsilon_0\hbar} \cdot \frac{(\rho_{11}-\rho_{22})}{i\Delta + A_{12}}$$

Note that real & imaginary part modify propagation of field according to:

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left(\frac{-\partial}{\partial t} + \frac{1}{c}\frac{\partial}{\partial t}\right)E = -\mu_0 \frac{\partial^2}{\partial t^2}P(z,t)$$

slowly varying envelope approx. $\left(-\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)E \simeq 2ikE + \underbrace{\left(\frac{\partial}{\partial z}E\right)}_{\text{neglect.}}$

(corresponds to neglecting $\frac{\partial^2}{\partial z^2}$ term) $\quad E(z,t) = \mathcal{E}(z,t)e^{i\omega t - k\vec{r}i} \quad k = 2\pi/\lambda$

$$\frac{\partial \mathcal{E}}{\partial z} + \frac{1}{c}\frac{\partial \mathcal{E}}{\partial t} = -\frac{1}{2\varepsilon_0}k \cdot \mathrm{Im}\{\vec{P}\} = \underbrace{-\frac{1}{2\varepsilon_0}k \cdot \varepsilon_0 \chi''_{img}}_{\text{absorption or gain}} \cdot \mathcal{E}(z,t)$$

$$\frac{\partial \Phi}{\partial z} + \frac{1}{c}\frac{d\Phi}{dt} = \underbrace{\left(k - \frac{\omega}{c}\right)}_{\text{linear dispersion relation}} - \underbrace{\frac{1}{2\varepsilon_0}k \cdot \frac{1}{\mathcal{E}} \cdot \mathrm{Re}\{P\}}_{\hat{=}\frac{1}{\mathcal{E}}\frac{1}{2\varepsilon_0}k \cdot \chi_{Re}(\omega)\varepsilon_0\mathcal{E}}$$

Thus $\quad \left(k - \frac{\omega}{c}\right) - \frac{k \cdot \chi_{RE}(\omega)}{2} = \frac{\partial \Phi}{\partial z} + \frac{1}{c}\frac{d\Phi}{dt}$

new dispersion relation: (constant phase condition) $\partial_t \Phi = \partial_z \Phi = 0$

$$\boxed{\begin{aligned} k &= \left(1 + \frac{\chi_{Re}}{2}\right)\frac{\omega}{c} = n'\frac{\omega}{c} \\ n &= n' + in'' \;,\; n'' = \frac{\chi_{Im}}{2} \end{aligned}}$$

$$\boxed{n(\omega) \hat{\simeq} 1 + \frac{\chi_{RE}}{2}} \quad \text{if } |\chi_{Re}| \gg |\chi_{Im}|$$

refractive index.

$$\boxed{\begin{aligned} n \simeq n' &= \frac{c}{v_p} = \frac{ck}{\omega} \\ n^2(\omega) &= 1 + \chi \end{aligned}}$$

gain coefficient

$$\boxed{g(\omega) = -\frac{1}{2}k \cdot \chi''_{img}(\omega)}$$

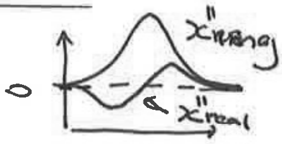
(note unperturbed solution: $\Phi = (kz - \frac{\omega}{c}t)$)

note:

$$\chi'_{real} = N\frac{\vec{p}_{12}^{\,2}}{\varepsilon_0\hbar}\frac{\Delta}{A_{12}^2 + \Delta^2}(\rho_{11} - \rho_{22})$$

$$\chi''_{img} = N\frac{\vec{p}_{12}^{\,2}}{\varepsilon_0\hbar}\frac{A_{12}}{A_{12} + \Delta^2}(\rho_{11} - \rho_{22})$$

Quantum Mechanical model for $n(\omega)$



$(\rho_{11}-\rho_{22})>0$ case

Note can be generalized to multiple atoms

ie $\chi = \frac{P}{\varepsilon_0\mathcal{E}} = 2\left[\frac{\rho_{12}p_{12}}{\varepsilon_0\mathcal{E}} + \frac{\rho_{13}p_{13}}{\varepsilon_0\mathcal{E}}\right]e^{i\omega t}$ multi-level susceptibility.

→ e.g. allows achieving high n with low absorption

→ SLOW LIGHT effects.

→ LAMB's semiclassical laser theory

Quantum Optics 1
-Lecture 7-
(Quantum Atom-Field interaction)

RWA approximation:

$$\boxed{\hat{H}_{inl} = \hbar\omega_k \hat{a}_k^+ \hat{a}_k + \frac{1}{2}\hbar\omega_{12}\hat{\sigma}_z + \hbar \cdot \sum_k g_k(\hat{\sigma}_+ \hat{a}_k^+ + \hat{\sigma}_- \hat{a}_k)}$$

Derive the equation of motion.

coupling coefficient $$\boxed{g_k = \frac{\hat{\varepsilon}_k \cdot \overset{\text{ppf}}{\mathcal{E}} \vec{p}_{12}}{\hbar} = \frac{\vec{p}_{12} \cdot \vec{\varepsilon}_k \cdot}{\hbar}\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V_k}}}$$

Compute the Rabi frequency. Note that in the quantum mechanical model it becomes photon-number dependent.

$$\langle 1, n | \hbar g_k \hat{\sigma}^- \hat{a}_k^+ | 2, n-1\rangle = \underbrace{\langle 1 | \hat{\sigma}^- | 2\rangle}_{1} \cdot \underbrace{\langle n | \hat{a}_k^+ | n-1\rangle}_{n | n \rangle} g_k \hbar$$

$$= \sqrt{n}\, g_k \hbar \qquad a_k^+ |n\rangle = \sqrt{n+1}\,|n+1\rangle$$

analogous:

$$\langle 2, n-1 | \hbar g_k \hat{\sigma}^+ \hat{a}_k | 1, n\rangle = \underbrace{\langle 2 | \sigma^+ | 1\rangle}_{1} \langle n-1 | \underbrace{\hat{a}_k | n\rangle}_{\hat{=} \sqrt{n} \cdot |n-1\rangle} \hbar g_k$$

$$= \sqrt{n}\, g_k \hbar \quad \rightarrow \text{photon number dependent}$$

Compute evolution in the manifold of:

$$|\psi(t)\rangle = |1\rangle |n+1\rangle c_1(t) + |2\rangle |n\rangle c_2(t)$$

(thus $\sqrt{n+1}\, g_k \hbar$ are coupling terms)

$$\Rightarrow \boxed{\begin{aligned} \frac{d}{dt}c_1 &= -ig\sqrt{n+1}\; c_2\, e^{+i(\omega - \omega)t} \\ \frac{d}{dt}c_2 &= -ig\sqrt{n+1}\; c_1\, e^{-i(\omega - \omega)t} \end{aligned}} \quad *$$

As in the semiclassical case we obtain Rabi-oscillations but now with $\boxed{\Omega_n^2 = \Delta^2 + 4g^2(n+1)}$

* Note: here we transformed to <u>INTERACTION PICTURE</u>:

recall $i\hbar\partial_t|\psi\rangle = \hat{H}_0|\psi(t)\rangle \qquad |\psi(t)\rangle = |\psi(0)\rangle e^{-i\hat{H}_0/\hbar t}$

Thus: $$\langle\psi|\hat{a}|\psi\rangle = \langle\psi_0| \underbrace{e^{+i\hat{H}_0/\hbar t}\, \hat{a}\, e^{-i\hat{H}_0/\hbar t}}_{\text{operator transformation}} |\psi(0)\rangle$$

$$e^{i\omega_c \hat{a}^+\hat{a} t}\, \hat{a}\, e^{-i\omega_c a^+ a t} = a e^{-i\omega_c t} \quad \text{using } e^{\alpha\hat{A}}\hat{B}e^{-\alpha\hat{A}} = \hat{B} + \alpha[\hat{A},\hat{B}] + \alpha^2[A,[A,B]] \ldots$$

$$e^{i\omega_{12}\hat{\sigma}_z t/2}\,\hat{\sigma}_+\, e^{-i\omega_{12}\hat{\sigma}_z t/2} = \hat{\sigma}_+ e^{i\omega_{12} t}$$

$$\Rightarrow \boxed{\hat{H}_{int}^{\text{interaction}} = \hbar g \hat{\sigma}_+ \hat{a}\, e^{-i(\omega - \omega)t} + \hbar g \hat{\sigma}_- \hat{a}_k^+ e^{+i(\omega - \omega)t}}$$

Quantum Optics Lecture 7

Eigenstates of atom field interaction. Diagonal Basis

$$|\Phi^+\rangle = \frac{1}{\sqrt{2}}(|n,2\rangle + |n+1,1\rangle)$$

$$|\Phi^-\rangle = \frac{1}{\sqrt{2}}\,|n,2\rangle - |n+1,1\rangle)$$

} entangled states of atom and field.

Spontaneous emission : WIGNER-WEISSKOPF THEORY

assume that $|\psi(t)\rangle = \sum_k \underbrace{|0\rangle_k}_{\text{field}} \otimes \underbrace{|2\rangle}_{\text{atom}}$ at $t=0$ $\quad$ $|0_k\rangle$: vacuum of mode k

$$|\psi(t)\rangle = c_1|0_k, 2\rangle + \underbrace{\sum_k c_2^k |1_k, 1\rangle}_{\text{all possible modes of the field.}} \qquad |0\rangle \equiv |0_1\rangle|0_2\rangle \ldots |0_k\rangle$$

Thus we obtain

$$\left|\begin{array}{l} \dot{c}_1 = -i\sum_k g_k e^{+i(\omega_{12}-\omega_k)t} c_{2,k} \\ \dot{c}_{2,k} = i g_k e^{-i(\omega_{12}-\omega_k)t} c_1 \end{array}\right|$$

equation for wavefunction expansion coefficient ie prob amplitudes.

eliminate $c_{2,k}$: $\quad c_{2,k} = -ig(r_0)\cdot\int_0^t dt'\, e^{-i(\omega_{12}-\omega_k)t} c_1(t')$

$$\boxed{\frac{d}{dt}c_1 = \dot{c}_1 = \sum_k |g_k(r)|^2 \cdot \int_0^t dt'\, e^{i(\omega_{12}-\omega_k)(t-t')} c_1(t')}$$

Next introduce the DENSITY OF STATES. Note $\omega = k.c$; $dk = d\omega/c$

$$\sum_k \rightarrow \underset{\uparrow\ \text{Pol.}}{2}\cdot\frac{V}{(2\pi)^3}\int_0^{2\pi} d\Phi \int_0^{\pi} d\theta \sin\theta \int dk\, k^2 = 2$$

$$= 2\cdot\frac{V}{8\pi^3}\frac{1}{c^3}\int_0^{2\pi} d\Phi \int_0^{\pi} d\theta \sin\theta \int d\omega_k\, \omega_k^2$$

Thus:

Note that $\boxed{|g(\vec{r}_0)|^2 = \left(\sqrt{\frac{\hbar\omega_k}{2\varepsilon_0 V}}\right)^2 |p_{12}|^2\cdot\cos^2\theta}$ $\quad$ where $\cos\theta$ is angle between $\vec{E}_k$ and $\vec{p}_{12}$

ie $\varepsilon_k\cdot p_{12} = |p_{12}|\cdot\cos\theta$

Hence one obtains: (eliminating integral over θ, Φ): $\int_0^{\pi}\cos^2\theta\sin\theta\, d\theta = \left[\frac{\cos^3\theta}{3}\right]_0^{\pi} = \frac{1}{3}$

$$\frac{d}{dt}c_1 = -\frac{4|\vec{p}_{12}|^2}{(2\pi)^2\, 6\hbar\varepsilon_0 c^3}\int_0^{\infty} d\omega_k\, \omega_k^3 \int_0^t dt'\, e^{i(\omega_{12}-\omega)t} \underset{\wedge}{c}(t')$$

I) let $\omega_k \rightarrow -\infty$

II) assume that $\omega_k^3 \simeq \omega^3$ Pull out of integral.

Note that $\int_{-\infty}^{\infty} d\omega_k\, e^{-i(\omega_{12}-\omega_k)(t-t')} = 2\pi\,\delta(t-t')$

For $\int_{-\infty}^{t} \delta(t-t')\,c(t')\,dt' = c(t)/2$ $\quad$ (ie "half the integral $\int_{-\infty}^{\infty}\delta(t-t')\,dt'\,c(t') = c(t)$

III). δ-function integration

$$\Rightarrow \boxed{\frac{d}{dt}c_1(t) = -\underbrace{\left[\frac{1}{4\pi\varepsilon_0}\frac{4\omega^3|\vec{p}_{12}|^2}{3\hbar c^3}\right]}_{\hat{=}\,\Gamma}\frac{1}{2}\,c_1(t)}$$

Spontaneous emission rate QUANTUM MECH.